

# Dynamics of the diffusive DM-DE interaction–dynamical system approach

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**Abstract.** We discuss dynamics of a model of an energy transfer between dark energy (DE) and dark matter (DM). The energy transfer is determined by a non-conservation law resulting from a diffusion of dark matter in an environment of dark energy. The relativistic invariance defines the diffusion in a unique way. The system can contain baryonic matter and radiation which do not interact with the dark sector. We treat the Friedman equation and the conservation laws as a closed dynamical system. The dynamics of the model is examined using the dynamical systems methods for demonstration how solutions depend on initial conditions. We also fit the model parameters using astronomical observation: SNIa,  $H(z)$ , BAO and Alcock-Paczynski test. We show that the model with diffuse DM-DE is consistent with the data.

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## 1 Introduction

In spite of an excellent agreement of the  $\Lambda$ CDM model with observational data some basic assumptions of this model need justification. There are some ingredients in the model which could hardly be derived from a certain fundamental theory. The presence of dark energy (DE) with its currently small value is difficult to explain in the standard model of elementary particles [1]. Then, the relation of dark energy to the dark matter (DM) seems accidental (coincidence problem). That these components are of the same order suggests that there may be certain dynamical relation between them. We suggest a model describing an irreversible flow of DE to DM. We assume that the total mass of the dark matter does not change. These assumptions lead to the unique model of the DM-DE interaction.

The Einstein equations are

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}, \quad (1.1)$$

where  $R^{\mu\nu}$  is the Ricci tensor,  $g^{\mu\nu}$  the metric and  $8\pi G = c = \hbar = 1$ , we can decompose the right-hand sides of (1.1) as

$$T^{\mu\nu} = T_b^{\mu\nu} + T_R^{\mu\nu} + T_{de}^{\mu\nu} + T_{dm}^{\mu\nu}, \quad (1.2)$$

where the absence of an interaction between baryonic matter  $T_b$ , radiation  $T_R$  and the dark component means

$$\nabla_\mu(T_R^{\mu\nu} + T_b^{\mu\nu}) = 0. \quad (1.3)$$

The conservation of the total energy gives

$$\nabla_\mu T_{de}^{\mu\nu} = -\nabla_\mu T_{dm}^{\mu\nu} \equiv -3\kappa^2 J^\nu \quad (1.4)$$

with a current  $J^\nu$  and a certain constant  $\kappa$  which can be calculated when the model of  $T_{dm}^{\mu\nu}$  is defined.

The relation between the non-conservation law (1.4) can explain the coincidence between DM and DE densities as well as the relevance of the dark energy exactly at the present epoch. We need a model for  $T_{de}^{\mu\nu}$  and  $T_{dm}^{\mu\nu}$ . We assume that the gain of energy of the dark matter consisting of particles of mass  $m$  results from a diffusion in an environment described by an ideal fluid. There is only one diffusion which is relativistic invariant and preserves the particle mass  $m$  [2]. The corresponding energy-momentum satisfies the conservation law (1.4). The current  $J^\nu$  in eq. (1.4) is conserved [3–5]

$$\nabla_\mu J^\mu = 0. \quad (1.5)$$

This is a realization of the conservation law of the total mass of the dark matter. In a homogeneous universe the current conservation implies

$$J^0 = \frac{\gamma}{3\kappa^2} a^{-3}, \quad (1.6)$$

where  $a$  is the scale factor of an expanding metric and another constant  $\gamma$ .

In a homogeneous space-time we can represent the DM as well as DE energy-momentum as the energy-momentum of an ideal fluid. The conservation law (1.4) leads to a particular interaction among the fluids. An interaction which is a linear combination of the DM and DE fluids has been discussed in [6]. Non-linear interactions are discussed in [7–10]. Our formula for the DM dissipation (1.4) follows from the assumption that the dissipation results from a relativistic motion in a DE fluid. It cannot be expressed as a polynomial formula in DM and DE fluids as it is in the above mentioned references. Nevertheless, we are able to express the dynamics of the model as a quadratic dynamical system what makes our approach similar to that of refs. [7, 8, 10].

Methods of dynamical systems [11] have been recently used in a cosmological model with diffusion described by a cosmological scalar field [12]. A similar analysis of the dynamics has been also explored in the context of Bianchi cosmological models [13] as well as in a description of non-homogeneous and anisotropic cosmological models [14]. In this paper we intend to explore cosmological models as closed dynamical systems with matter (dark and baryonic) and dark energy in the form of ideal fluids whose interaction is determined by the current  $J^\nu$  (1.4). In contradistinction to above mentioned models our model does not contain non-physical trajectories passing through  $\rho_m = 0$  line [15].

The plan of the paper is the following. In sec. 2 we review the model of a relativistic diffusion and explain eq. (1.4). In sec. 3 we derive exactly soluble limits relevant for early and late universe. We discuss energy-momentum conservation and Einstein equations in sec. 4. In sec. 5 we formulate the cosmological equations of sec. 4 as a closed dynamical system. We determine its critical points and the phase portrait. In sec. 6 we fit the parameters of the model to the observational data.

## 2 Relativistic diffusion

In this section we consider a Markovian approximation of an interaction of the system with an environment which leads to the description of this interaction by a diffusion. We consider

a relativistic generalization of the Krammers diffusion defined on the phase space. It is determined in the unique way by the requirement that the diffusing particle moves on the mass-shell (see [2, 16–18]).

Let us choose the contravariant spatial coordinates  $p^j$  on the mass shell and define the Riemannian metric

$$ds^2 = g_{\mu\nu} dp^\mu dp^\nu = -G_{jk} dp^j dp^k,$$

where Greek indices  $\mu, \nu = 0, 1, 2, 3$ , Latin indices  $j, k = 1, 2, 3$  and  $p_0$  is expressed by  $p^j$  from  $p^2 = m^2$ . We have (we assumed that  $g_{0k} = 0$ )

$$G_{jk} = -g_{jk} + p_j p_k \omega^{-2},$$

where

$$\omega^2 = m^2 - g_{jk} p^j p^k.$$

Then, the inverse matrix is

$$G^{jk} = -g^{jk} + m^{-2} p^j p^k.$$

Next,

$$G \equiv -\det(G_{jk}) = -m^2 \det(g_{jk}) \omega^{-2}.$$

We define diffusion as a stochastic process generated by the Laplace-Beltrami operator  $\Delta_H^m$  on the mass shell

$$\Delta_H^m = \frac{1}{\sqrt{G}} \partial_j G^{jk} \sqrt{G} \partial_k, \quad (2.1)$$

where  $\partial_j = \frac{\partial}{\partial p^j}$  and  $G = \det(G_{jk})$  is the determinant of  $G_{jk}$ .

The transport equation for the diffusion generated by  $\Delta_H$  reads

$$(p^\mu \partial_\mu^x - \Gamma_{\mu\nu}^k p^\mu p^\nu \partial_k) \Omega = \kappa^2 \Delta_H^m \Omega, \quad (2.2)$$

where  $\Gamma_{\mu\nu}^k$  are the Christoffel symbols,  $\partial_\mu^x$  are space-time derivatives and  $\kappa^2$  is the diffusion constant.

Then, we can define the current

$$J^\mu = \sqrt{g} \int \frac{d\mathbf{p}}{(2\pi)^3} p_0^{-1} p^\mu \Omega, \quad (2.3)$$

where  $g = |\det[g_{\mu\nu}]|$  and the energy momentum

$$T_{de}^{\mu\nu} = \sqrt{g} \int \frac{d\mathbf{p}}{(2\pi)^3} p_0^{-1} p^\mu p^\nu \Omega. \quad (2.4)$$

It can be shown using eq. (2.2) that [3–5]

$$\nabla_\mu T_{\text{dm}}^{\mu\nu} = 3\kappa^2 J^\nu \quad (2.5)$$

and

$$\nabla_\mu J^\mu = g^{-\frac{1}{2}} \partial_\mu (g^{\frac{1}{2}} J^\mu) = 0. \quad (2.6)$$

Hence,

$$g^{-\frac{1}{2}} \partial_t (g^{\frac{1}{2}} J^0) = -\partial_j J^j. \quad (2.7)$$

This implies (1.6) if the metric is homogeneous and  $\Omega$  does not depend on  $x$ . The constant  $\gamma$  can be expressed from eq. (2.3) as

$$\frac{\gamma}{3\kappa^2} = g \int \frac{d\mathbf{p}}{(2\pi)^3} \Omega \equiv Z. \quad (2.8)$$

### 3 The limits $ma \rightarrow 0$ and $ma \rightarrow \infty$

Most of our subsequent results hold true for a general FWR metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2 h_{jk} dx^j dx^k, \quad (3.1)$$

but for simplicity of our analysis we restrict ourselves to the flat space  $h_{jk} = \delta_{jk}$ . We rewrite the diffusion equation in terms of the covariant momenta

$$q_j = g_{ij} p^j \quad (3.2)$$

Then,  $\Delta_H^m$  in eq. (2.1) depends on  $\sqrt{m^2 a^2 + \mathbf{q}^2}$  and on  $a$ . The assumption  $\mathbf{q}^2 \gg m^2 a^2$  (high energy approximation) is equivalent to the limit

$$m^2 a^2 \rightarrow 0. \quad (3.3)$$

Let

$$\nu = \int dt a \quad (3.4)$$

Then, in the limit  $m^2 a^2 \rightarrow 0$  and in a homogeneous universe ( $\Omega$  independent of spatial coordinates) we obtain

$$\kappa^{-2} |\mathbf{q}| \partial_\nu \Omega = q_i q_j \frac{\partial^2}{\partial q_i \partial q_j} \Omega + 3 q_j \frac{\partial}{\partial q_j} \Omega \quad (3.5)$$

or in the original contravariant coordinates

$$a \kappa^{-2} |\mathbf{p}| (\partial_t - 2H p^j \frac{\partial}{\partial p^j}) \Omega = p^i p^j \frac{\partial^2}{\partial p^i \partial p^j} \Omega + 3 p^j \frac{\partial}{\partial p^j} \Omega. \quad (3.6)$$

If in  $\sqrt{m^2 a^2 + \mathbf{q}^2}$  we assume  $\mathbf{q}^2 \ll m^2 a^2$  (low energy approximation, i.e., we neglect  $\mathbf{q}$ ) then in the limit

$$m^2 a^2 \rightarrow \infty$$

eq. (2.2) simplifies to

$$m^{-1} \kappa^{-2} \partial_\sigma \Omega = \frac{1}{2} \Delta_{\mathbf{q}} \Omega, \quad (3.7)$$

where

$$\sigma = 2 \int_{t_0}^t ds a^2 \quad (3.8)$$

and  $\Delta_{\mathbf{q}}$  is the Laplacian. This is the non-relativistic diffusion equation. In terms of the original contravariant momenta eq. (3.7) takes the form

$$m^{-1} a^2 \kappa^{-2} \left( \partial_t - 2H p^j \frac{\partial}{\partial p^j} \right) \Omega = \frac{1}{2} \Delta_{\mathbf{p}} \Omega. \quad (3.9)$$

## 4 Current conservation and Einstein equations

The energy-momentum (2.4) in a homogeneous space-time can be expressed as an energy-momentum of a fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - g^{\mu\nu}p, \quad (4.1)$$

where

$$g_{\mu\nu}u^\mu u^\nu = 1. \quad (4.2)$$

The divergence equations (1.4) in the frame  $u = (1, \mathbf{0})$  takes the form

$$\partial_t \rho_{dm} + 3H(1 + \tilde{w})\rho_{dm} = \gamma a^{-3}, \quad (4.3)$$

where

$$\tilde{w} = \frac{p_{dm}}{\rho_{dm}}.$$

We assume that the energy-momentum tensor of the dark energy has also the form of an ideal fluid (4.1). Then, from eqs. (1.4) and (1.6)

$$\partial_t \rho_{de} + 3H(1 + w)\rho_{de} = -\gamma a^{-3}. \quad (4.4)$$

On the basis of observational data we choose  $w = -1$  in eq. (4.4). In the diffusion model  $\tilde{w}$  depends on time as follows from the formula

$$\begin{aligned} \tilde{w} &= \frac{1}{3} \int d\mathbf{p} \frac{1}{p_0} a^2 \mathbf{p}^2 \Omega_t \left( \int d\mathbf{p} p_0 \Omega_t \right)^{-1} \\ &= \frac{1}{3} - \frac{m^2}{3} \int d\mathbf{p} \frac{1}{p_0} \Omega_t \left( \int d\mathbf{p} p_0 \Omega_t \right)^{-1} \equiv \frac{1-\omega}{3}. \end{aligned}$$

In an expansion in  $m$  we can apply the explicit solution [19] of eq. (3.6) ( $m = 0$ ) and calculate

$$\omega = \frac{m^2 a^2}{6(T_0 + \kappa^2 \nu)},$$

where  $T_0$  is a parameter which has the meaning of the temperature of the DM fluid at  $t = t_0$ . In the ultrarelativistic (massless) case (3.6) we have  $\omega = 0$ , hence  $\tilde{w} = \frac{1}{3}$ .

We can express the solution of eq. (4.3) as

$$\begin{aligned} \rho_{dm}(t) &= \rho_{dm}(0) a^{-4} \exp \left( \int_{t_0}^t d\tau H \omega \right) \\ &\quad + \gamma a^{-4} \exp \left( \int_{t_0}^t d\tau H \omega \right) \int_{t_0}^t ds a(s) \exp \left( - \int_{t_0}^s d\tau H \omega \right). \end{aligned} \quad (4.5)$$

For  $w = -1$

$$\rho_{de}(t) = \rho_{de}(0) - \gamma \int_{t_0}^t a^{-3}(s) ds. \quad (4.6)$$

We still consider the non-relativistic limit of the energy-momentum (2.4)

$$\begin{aligned} \rho_{dm} = \tilde{T}^{00} &= \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} p^0 \Omega = g^{-\frac{1}{2}} Z m + \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \frac{a^2 \mathbf{p}^2}{2m} \Omega \\ &\equiv Z m a^{-3} + a^{-2} \rho_{nr}, \end{aligned} \quad (4.7)$$

where

$$\rho_{nr} = \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \Omega a^4 \frac{\mathbf{p}^2}{2m}. \quad (4.8)$$

Using the non-relativistic diffusion equation (3.9) we can show that  $\rho_{nr}$  satisfies the non-conservation equation

$$\partial_t a^{-2} \rho_{nr} + 3H(1 + \tilde{w}_{nr})a^{-2} \rho_{nr} = \gamma a^{-3}, \quad (4.9)$$

where  $\tilde{w}_{nr} = \frac{2}{3}$ . The non-relativistic diffusive energy in eq. (4.7) is a sum of two terms  $Zma^{-3}$  which describes a conservative non-relativistic total rest mass and  $a^{-2}\rho_{nr}$  describing the diffusive energy gained from the motion in an environment of the dark energy. The sum of these energies satisfies the equation (which is not of the form) (1.4))

$$\partial_t \rho_{dm} + 5H\rho_{dm} = 3Z\kappa^2 a^{-3} + 2ZmHa^{-3}. \quad (4.10)$$

The solution of eq. (4.9) is

$$\tilde{\rho}_{nr}(t) = a^{-3}(\rho_{nr}(0) + \frac{1}{2}\gamma\sigma), \quad (4.11)$$

where  $\sigma$  is defined in eq. (3.8). As a consequence of eqs. (4.7), (4.9) and (1.4) the non-relativistic dark energy satisfies the same eqs. (4.4) and (4.6) as the relativistic dark energy.

The Friedman equation in the FRW metric (3.1) with the dark matter, dark energy and baryonic matter  $\rho_b$  reads

$$H^2 = \frac{1}{3}(\rho_{dm} + \rho_{de} + \rho_b). \quad (4.12)$$

By differentiation

$$\dot{H} = -\frac{1}{2}\left((1 + \tilde{w})\rho_{dm} + \rho_b\right). \quad (4.13)$$

In eqs. (4.12)-(4.13) we should insert the general expressions for DM and DE. We need an approximation for  $\tilde{w}(t)$ . There we shall discuss approximations to eq. (4.5). In a subsequent section we study the relativistic homogeneous dynamical system (4.3), (4.4) and (4.13) under the assumption that  $\tilde{w}$  is time independent. The non-relativistic (low  $z$ ) approximation (4.7) when inserted in eq. (4.12) gives the Friedmann equation

$$H^2 = \frac{1}{3}\left(a^{-5}(\rho_{dm}(0) + \frac{3}{2}Z\kappa^2\sigma) + Zma^{-3} + \rho_b(0)a^{-3} + \rho_{de}(0) - 3Z\kappa^2 \int_{t_0}^t ds a(s)^{-3}\right). \quad (4.14)$$

Eqs. (4.6), (4.10) and (4.14) form a system of ordinary differential equations which is expressed by means of new (energetic) variables into a quadratic dynamical system in the next section.

## 5 Dynamical system approach to the DM-DE interaction

In this section we reduce the dynamics of the diffusive DM-DE interaction to the form of autonomous dynamical system  $\frac{dx}{dt} = \dot{x} = f(x)$ , where  $x$  is a state variable and  $t$  is time. In this approach one describes the evolution of the diffusive DM-DE interaction in terms of trajectories situated in a space of all states of the system, i.e., a phase space. This space

possesses the geometric structure which is a visualization of a global dynamics, i.e. it is the space of all evolutionary paths of the physical system, which are admissible for all initial conditions. The equivalence of phase portraits is established by means of a homeomorphism (topological equivalence) which is mapping trajectories of the system while preserving their orientation. The phase space is organized by critical points, which from the physical point of view represent stationary states of the system. From the mathematical point of view they are singular solutions of the system  $\dot{\mathbf{x}} = f(\mathbf{x})$ , where  $\mathbf{x} \in \mathbb{R}^n$  is a vector state, corresponding to vanishing right-hand sides of the system, i.e.  $\mathbf{f} = [f^1(x), \dots, f^n(x)]$  and  $\forall_i f^i(x) = 0$ . The final outcome of any dynamical system analysis is the phase portrait of the system from which one can easily obtain the information about the stability and genericity of particular solutions.

The methods of dynamical systems [11], which enable us to investigate the dynamics of the system without the knowledge of its exact solutions, have been recently applied in a similar context of cosmological models with diffusion [12]. An analysis of cosmological dynamics has also been explored in Bianchi cosmological models [13]. Some of these methods are applicable to non-homogeneous and anisotropic cosmological models [14] as well. In this paper we intend to explore an energy exchange in models describing matter (dark and baryonic) and dark energy in the form of the cosmological ideal fluids. In contrast to Alho et al. [12] our model does not contain non-physical trajectories passing through  $\rho_m = 0$  line [15].

### 5.1 Cosmological models with constant equation of state for DM and cosmological constant—dynamical system analysis

Let us consider the continuity equations for the model with  $\tilde{w} = \text{const}$  and  $w = -1$  (dark energy in the form of the cosmological constant). The corresponding continuity equations take the form (4.3)-(4.4)

$$a^{-3(\tilde{w}+1)} \frac{d}{dt} (\rho_{\text{dm}} a^{3(\tilde{w}+1)}) = \gamma a^{-3} > 0, \quad (5.1)$$

$$\frac{d\rho_{\text{de}}}{dt} = -\gamma a^{-3} < 0, \quad (5.2)$$

where  $\gamma > 0$ . To formulate the dynamics in the form of a dynamical system, we rewrite in a suitable way equation (5.1)

$$J = \frac{d\rho_{\text{dm}}/\rho_{\text{dm}}}{da/a} \equiv \frac{d \ln \rho_{\text{dm}}}{d(\ln a)} = -3(1 + \tilde{w}) + \frac{\gamma a^{-3}}{H \rho_{\text{dm}}}. \quad (5.3)$$

Next we define a dimensionless quantity, which measures the strength of the interaction

$$\delta \equiv \frac{\gamma a^{-3}}{H \rho_{\text{dm}}}. \quad (5.4)$$

Clearly, in general  $\delta$  is time dependent. Let us consider that  $\delta = \delta(a(t))$ . If this quantity is constant during the cosmic evolution, then the solution of eq. (5.3) has a simple form

$$\rho_{\text{dm}} = \rho_{\text{dm},0} a^{-3(1+\tilde{w})+\delta}. \quad (5.5)$$

Our aim is to study the dynamics of the energy transfer from the DE to DM sector. The corresponding system assumes the form of a three-dimensional dynamical system.



Recalling that  $8\pi G = c = 1$  we define

$$x \equiv \frac{\rho_{\text{dm}}}{3H^2}, \quad y \equiv \frac{\rho_{\Lambda}}{3H^2}, \quad (5.6)$$

where  $H = \frac{d \ln a}{dt}$  is the Hubble parameter and  $t$  is the cosmological time. The differentiation with respect to the cosmological time  $t$  will be denoted by a dot ( $\dot{\phantom{x}} \equiv \frac{d}{dt}$ ). The variables  $x$  and  $y$  have the meaning of dimensionless density parameters.

For simplicity of presentation it is assumed that FRW space is flat (zero curvature in the Friedmann equations (4.12)). In this case the acceleration equation assumes the following form

$$\dot{H} = -\frac{1}{2}(\rho_{\text{eff}} + p_{\text{eff}}), \quad (5.7)$$

where  $\rho_{\text{eff}} = \rho_{\text{dm}} + \rho_{\text{de}}$  and  $p_{\text{eff}} = \tilde{w}\rho_{\text{dm}} - \rho_{\text{de}}$  are the effective energy density and pressure of the matter filling the universe,  $\tilde{w} = \frac{p_{\text{dm}}}{\rho_{\text{dm}}}$ .

Taking a natural logarithm of the state variables (5.6) and the interaction effect variable (5.4) and performing the differentiation with respect to the cosmological time  $t$  we obtain

$$\frac{\dot{x}}{x} = \frac{\dot{\rho}_{\text{dm}}}{\rho_{\text{dm}}} - 2\frac{\dot{H}}{H} = -3H(1 + \tilde{w}) + \delta H - 2\frac{\dot{H}}{H}, \quad (5.8)$$

$$\frac{\dot{y}}{y} = \frac{\dot{\rho}_{\Lambda}}{\rho_{\Lambda}} - 2\frac{\dot{H}}{H} = -\delta H \alpha - 2\frac{\dot{H}}{H}, \quad (5.9)$$

$$\frac{\dot{\delta}}{\delta} = -3H - \frac{\dot{H}}{H} - \frac{\dot{\rho}_{\text{dm}}}{\rho_{\text{dm}}} = 3\tilde{w}H - \delta H - \frac{\dot{H}}{H}, \quad (5.10)$$

where  $\alpha = \frac{\rho_{\text{dm}}}{\rho_{\Lambda}}$ .

It would be convenient to divide both sides of the system (5.8)-(5.10) by  $H$  and then reparameterize the original time variable  $t$  following the rule

$$t \rightarrow \tau = \ln a. \quad (5.11)$$

The differentiation with respect to the parameter  $\tau$  will be denoted by a prime ( $' \equiv \frac{d}{d\tau}$ ). Note that  $\frac{d\tau}{da} = a^{-1}$  is a strictly monotonic function of the scale factor  $a$ .

After the time reparameterization (5.11) the system (5.8)-5.10 can be expressed as the three-dimensional system of equations

$$x' = x \left( -3(1 + \tilde{w}) + \delta - 2\frac{\dot{H}}{H^2} \right), \quad (5.12)$$

$$y' = y \left( -\delta\alpha - 2\frac{\dot{H}}{H^2} \right), \quad (5.13)$$

$$\delta' = \delta \left( 3\tilde{w} - \delta - \frac{\dot{H}}{H^2} \right), \quad (5.14)$$

where  $\frac{\dot{H}}{H^2}$  can be determined from the formula (5.7)

$$\dot{H} = -\frac{1}{2}(1 + \tilde{w})\rho_{\text{dm}} = -\frac{3}{2}(1 + \tilde{w})H^2x, \quad (5.15)$$

i.e.,

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + \tilde{w})x. \quad (5.16)$$

In this way the dynamics of the process of decaying cold dark matter satisfying the equation of state  $p_{\text{dm}} = \tilde{w}\rho_{\text{dm}}$  in the background of the flat FRW metric can be described by means of the dynamical system theory. The resulting three-dimensional dynamical system has the form

$$x' = x(-3(1 + \tilde{w}) + \delta + 3(1 + \tilde{w})x), \quad (5.17)$$

$$y' = x(-\delta + 3(1 + \tilde{w})y), \quad (5.18)$$

$$\delta' = \delta \left( 3\tilde{w} - \delta + \frac{3}{2}(1 + \tilde{w})x \right), \quad (5.19)$$

where  $\alpha = x/y$ .

Note that the right-hand sides of the dynamical system (5.17)-(5.19) are of a polynomial form. Therefore all methods of dynamical system analysis, especially analysis of the behavior on the Poincaré sphere, can be adopted; both in a finite domain as well as at infinity. One can see that the system (5.17)-(5.19) has as an invariant submanifold  $\{\frac{\dot{H}}{H^2} = 0\}$ , the set  $\{x: x = 0\}$ , corresponding to the case of the vanishing dark matter energy density.

Clearly, the system (5.17)-(5.19) has also an invariant submanifold  $\{\delta: \delta = 0\}$  corresponding to the case of the vanishing interacting term  $Q = \delta$  as it appears in the  $\Lambda$ CDM model.

Another interesting submanifold is the plane  $\{y: y = \frac{\delta}{3(1+\tilde{w})}\}$ .

## 5.2 Dynamics of the model for dust matter

Let  $\tilde{w}$  be equal to zero. Then, the equation of state for matter is of the form of a dust. Because  $x + y = 1$  ( $\Omega_{\text{dm}} + \Omega_{\text{de}} = 1$ ) then the dynamical system (5.17)-(5.19) reduces to the two-dimensional dynamical system in the following form

$$x' = x(-3 + \delta + 3x), \quad (5.20)$$

$$\delta' = \delta \left( -\delta + \frac{3}{2}x \right). \quad (5.21)$$

The phase portrait for the dynamical system (5.20)-(5.21) is presented in Figure 1. On this phase portrait the deS<sub>+</sub> universe is a global attractor for expanding universes. On another hand the critical point (3) is a global repeller representing the Einstein-de Sitter universe. The saddle point is representing the static Einstein universe.

For the analysis of the behavior of trajectories at infinity we use the following sets of two projective coordinates:  $\tilde{x} = \frac{1}{x}$ ,  $\tilde{\delta} = \frac{\delta}{x}$  and  $\tilde{X} = \frac{x}{\delta}$ ,  $\tilde{\Delta} = \frac{1}{\delta}$ .

The dynamical system in variables  $\tilde{x}$  and  $\tilde{\delta}$  covers the behavior of trajectories at infinity

$$\tilde{x}' = \tilde{x}(3\tilde{x} - \tilde{\delta} - 3), \quad (5.22)$$

$$\tilde{\delta}' = \tilde{\delta} \left( 3\tilde{x} - 2\tilde{\delta} - \frac{3}{2} \right), \quad (5.23)$$

where  $' \equiv \tilde{x} \frac{d}{d\tau}$ . The phase portrait for the dynamical system (5.22)-(5.23) is presented in Figure 2.

**Table 1.** Critical points for autonomous dynamical systems (5.20)-(5.21), (5.22)-(5.23), (5.24)-(5.25), their type and cosmological interpretation.

No.	critical point	type of critical point	type of universe
1	$x = 0, \delta = 0$	saddle-node	de Sitter universe without diffusion effect
2	$x = 2/3, \delta = 1$ ( $\tilde{x} = 3/2, \tilde{u} = 3/2$ ) ( $\tilde{X} = 2/3, \tilde{U} = 1$ )	saddle	scaling universe
3	$x = 1, \delta = 0$ ( $\tilde{x} = 1, \tilde{u} = 0$ )	unstable node	Einstein-de Sitter universe without diffusion effect
4	$\tilde{x} = 0, \tilde{\delta} = 0$	stable node	static universe
5	$\tilde{X} = 0, \tilde{\Delta} = -3/4$	saddle	static universe
6	$\tilde{X} = 0, \tilde{\Delta} = 0$	unstable node	de Sitter universe with diffusion effect

The dynamical system for variables  $\tilde{X}$  and  $\tilde{\Delta}$  is described by the following equations

$$\tilde{X}' = \tilde{X} \left( -3\tilde{\Delta} + \frac{3}{2}\tilde{X} + 2 \right), \quad (5.24)$$

$$\tilde{\Delta}' = \tilde{\Delta} \left( 1 - \frac{3}{2}\tilde{X} \right), \quad (5.25)$$

where  $' \equiv \tilde{\Delta} \frac{d}{d\tilde{\tau}}$ . The phase portrait for the dynamical system (5.24)-(5.25) is presented in Figure 3.

We use also the Poincaré sphere to analyze critical points in the infinity. We define variables

$$X = \frac{x}{\sqrt{1+x^2+\delta^2}}, \quad \Delta = \frac{\delta}{\sqrt{1+\delta^2+x^2}} \quad (5.26)$$

and in these variables the dynamical system has the following form

$$X' = X \left[ -\Delta^2 \left( \frac{3}{2}X - \Delta \right) + (1 - X^2)(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (5.27)$$

$$\Delta' = \Delta \left[ (1 - \Delta^2) \left( \frac{3}{2}X - \Delta \right) - X^2(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (5.28)$$

where  $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{d\tilde{\tau}}$ . The phase portrait for the dynamical system (5.27)-(5.28) is presented in Figure 4. Critical points for autonomous dynamical systems (5.20)-(5.21), (5.22)-(5.23), (5.24)-(5.25) are completed in Table 1.

### 5.3 Dynamics of the model at the late time ( $ma \rightarrow \infty$ )

As can be seen from eq. (4.7) the relativistic model of the dark matter consists of two fluids first with  $\tilde{w} = 0$  and the second with  $\tilde{w} = \frac{2}{3}$ . So, in the approximation  $ma \rightarrow \infty$  and  $w = -1$  we have according to eqs. (4.6), (4.10) and (4.14) the following DM and DE continuity

equations

$$\dot{\rho}_{\text{dm}} + 5\rho_{\text{dm}}H = \gamma a^{-3} + 2ZmHa^{-3}, \quad (5.29)$$

$$\dot{\rho}_{\text{de}} = -\gamma a^{-3}. \quad (5.30)$$

We define the variables

$$x = \frac{\rho_{\text{dm}}}{3H^2}, \quad y = \frac{\rho_{\text{de}}}{3H^2}, \quad u = \frac{(2Zm)a^{-3}}{\rho_{\text{dm}}} \quad \text{and} \quad \delta = \frac{\gamma a^{-3}}{H\rho_{\text{dm}}}. \quad (5.31)$$

If we use the variables (5.31) and time  $\tau = \ln a$  then we obtain the following dynamical system

$$x' = x(-5 + \delta + u - 2\frac{\dot{H}}{H^2}), \quad (5.32)$$

$$y' = -x(\delta + u) - 2y\frac{\dot{H}}{H^2}, \quad (5.33)$$

$$u' = u(2 - \delta - u), \quad (5.34)$$

$$\delta' = \delta(2 - \delta - u - \frac{\dot{H}}{H^2}), \quad (5.35)$$

where  $' \equiv \frac{d}{d\tau}$  and  $\frac{\dot{H}}{H^2} = -\frac{1}{2}x(5 - u)$ . Because  $x + y = 1$  ( $\Omega_{\text{dm}} + \Omega_{\text{de}} = 1$ ) then the dynamical system (5.32)-(5.35) reduces to the three-dimensional dynamical system.

The dynamical system (5.32)-(5.35) has the invariant submanifold  $\{\frac{\dot{H}}{H^2} = 0\}$ , which is the set  $\{x: x = 0\}$  or  $\{u: u = 5\}$ . There is also an interesting submanifold  $\delta = 0$ . On the invariant submanifold  $\delta = 0$  the dynamical system (5.32)-(5.35) reduces to

$$x' = x(u + 5(x - 1) - xu), \quad (5.36)$$

$$u' = u(2 - u). \quad (5.37)$$

The phase portrait for the dynamical system (5.36)-(5.37) is presented in Figure 5. Note that critical point (1) is representing the deS<sub>+</sub> universe without the diffusion effect. On the other hand the de Sitter universe without diffusion is represented by saddle critical point. Therefore the model with diffusion is generic in the class of all trajectories.

For the analysis the behavior of trajectories at infinity we use the following two sets of projective coordinates:  $\tilde{x} = \frac{1}{x}$ ,  $\tilde{u} = \frac{u}{x}$  and  $\tilde{X} = \frac{x}{u}$ ,  $\tilde{U} = \frac{1}{u}$ .

The dynamical system for variables  $\tilde{x}$  and  $\tilde{u}$  is expressed by

$$\tilde{x}' = \tilde{x}(5\tilde{x}(\tilde{x} - 1) + \tilde{u}(1 - \tilde{x})), \quad (5.38)$$

$$\tilde{u}' = \tilde{u}(\tilde{x}(7\tilde{x} - 5) + \tilde{u}(1 - 2\tilde{x}) + \tilde{u}), \quad (5.39)$$

where  $' \equiv \tilde{x}^2 \frac{d}{d\tau}$ . The phase portrait for above dynamical system is presented in Figure 6. In comparison to the phase portrait in Figure 5 a new critical point (5) is emerging. It is representing the Einstein-de Sitter universe fully dominated by dark matter.

The dynamical system for variables  $\tilde{X}$  and  $\tilde{U}$  is described by the following equations

$$\tilde{X}' = \tilde{X}(\tilde{U}(2 - 7\tilde{U}) + \tilde{X}(5\tilde{U} - 1)), \quad (5.40)$$

$$\tilde{U}' = \tilde{U}^2(1 - 2\tilde{U}), \quad (5.41)$$

**Table 2.** Critical points for autonomous dynamical systems (5.36)-(5.37), (5.38)-(5.39), (5.40)-(5.41), their types and cosmological interpretations.

No	critical point	type of critical point	type of universe
1	$x = 0, u = 0$	saddle	de Sitter universe without diffusion effect
2	$x = 1, u = 2$ $(\tilde{x} = 1, \tilde{u} = 2)$ $(\tilde{X} = 1/2, \tilde{U} = 1/2)$	saddle	scaling universe
3	$x = 1, u = 0$ $(\tilde{x} = 1, \tilde{u} = 0)$	unstable node	Einstein-de Sitter universe without diffusion effect
4	$x = 0, u = 2$ $(\tilde{X} = 0, \tilde{U} = 1/2)$	stable node	de Sitter universe without diffusion effect
5	$\tilde{x} = 0, \tilde{u} = 0$	stable node	static universe
6	$\tilde{X} = 0, \tilde{U} = 0$	unstable node	de Sitter universe without diffusion effect

where  $' \equiv \tilde{U}^2 \frac{d}{d\tau}$ . The phase portrait for the dynamical system (5.40)-(5.41) is presented in Figure 7. Note the de Sitter universe represented by critical point (1) which is a stationary universe without effect of diffusion is a global attractor.

Critical points for autonomous dynamical systems (5.36)-(5.37), (5.38)-(5.39), (5.40)-(5.41) are completed in Table 2.

We apply also the Poincaré sphere to this system in order to analyze critical points at infinity. We define the variables

$$X = \frac{x}{\sqrt{1+x^2+u^2}}, \quad U = \frac{u}{\sqrt{1+x^2+u^2}} \quad (5.42)$$

In these variables the dynamical system has the following form

$$X' = X \left[ U^2 \sqrt{1-X^2-U^2} (U - 2\sqrt{1-X^2-U^2}) + (1-X^2)(\sqrt{1-X^2-U^2}(5X+U) - 5(1-X^2-U^2) - XU) \right], \quad (5.43)$$

$$U' = U \left[ (1-U^2)\sqrt{1-X^2-U^2}(2\sqrt{1-X^2-U^2} - U) - X^2(\sqrt{1-X^2-U^2}(5X+U) - 5(1-X^2-U^2) - XU) \right], \quad (5.44)$$

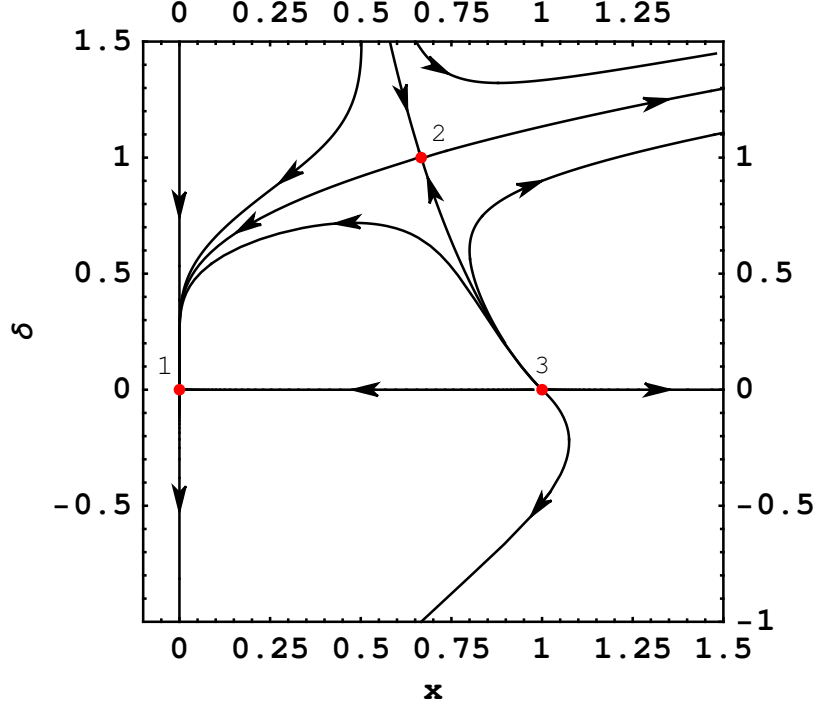
where  $' \equiv (1-X^2-U^2) \frac{d}{d\tau}$ . The phase portrait for the dynamical system (5.43)-(5.44) is presented in Figure 8.

## 6 Statistical analysis

### 6.1 Introduction

In this section we use astronomical observations for low redshifts such as the SNIa, BAO, measurements of  $H(z)$  for galaxies and the Alcock-Paczyński test. We do not use the observation for high redshifts such as CMB.

$$\ln L_{\text{SNIa}} = -\frac{1}{2}[A - B^2/C + \ln(C/(2\pi))], \quad (6.1)$$



**Figure 1.** A phase portrait for dynamical system (5.20)-(5.21). Critical point (1) ( $x = 0$ ,  $\delta = 0$ ) represents the de Sitter universe. Critical point (2) ( $x = 2/3$ ,  $\delta = 1$ ) is a saddle and represents the scaling universe. Critical point (3) ( $x = 1$ ,  $\delta = 0$ ) is an unstable node and represents the Einstein-de Sitter universe. The critical point (1) is a complex type of saddle-node.

where  $A = (\mu^{\text{obs}} - \mu^{\text{th}})\mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$ ,  $B = \mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$ ,  $C = \text{Tr}\mathbb{C}^{-1}$  and  $\mathbb{C}$  is a covariance matrix for SNIa. The distance modulus is expressed by  $\mu^{\text{obs}} = m - M$  (where  $m$  is the apparent magnitude and  $M$  is the absolute magnitude of SNIa) and  $\mu^{\text{th}} = 5 \log_{10} D_L + 25$  (where the luminosity distance is  $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$ ).

We also use BAO observations such as Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at  $z = 0.275$  [20], 6dF Galaxy Redshift Survey measurements at redshift  $z = 0.1$  [21], and WiggleZ measurements at redshift  $z = 0.44, 0.60, 0.73$  [22]. The likelihood function is expressed by the formula

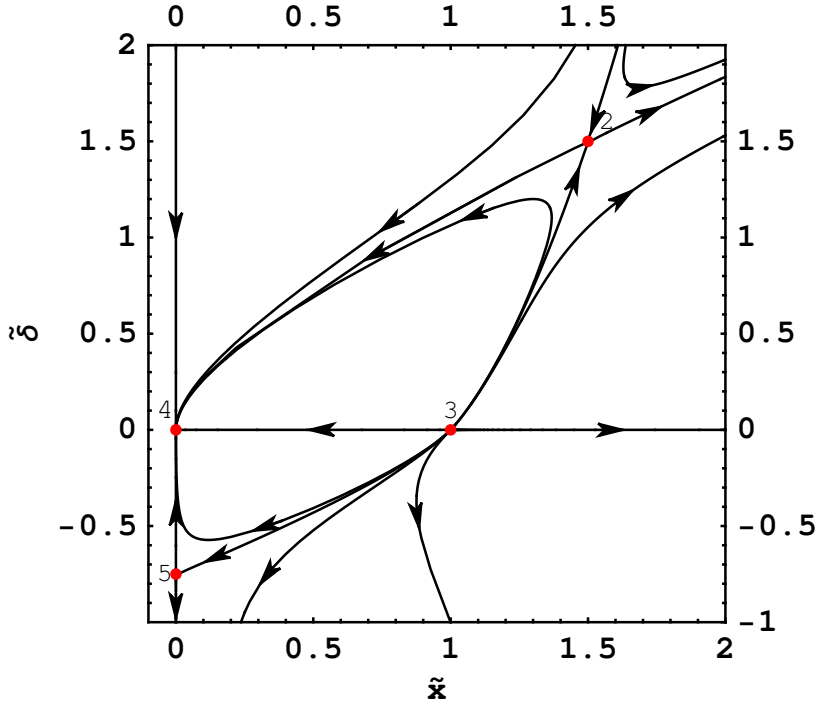
$$\ln L_{\text{BAO}} = -\frac{1}{2} \left( \mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \mathbb{C}^{-1} \left( \mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right), \quad (6.2)$$

where  $r_s(z_d)$  is the sound horizon at the drag epoch [23].

For the Alcock-Paczynski test [24, 25] we use the likelihood function

$$\ln L_{\text{AP}} = -\frac{1}{2} \sum_i \frac{(AP^{\text{th}}(z_i) - AP^{\text{obs}}(z_i))^2}{\sigma^2}. \quad (6.3)$$

where  $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$  and  $AP(z_i)^{\text{obs}}$  are observational data [26–34].



**Figure 2.** A phase portrait for dynamical system (5.22)-(5.23). Critical point (4) ( $\tilde{x} = 0, \tilde{\delta} = 0$ ) and (5) ( $\tilde{x} = 0, \tilde{\delta} = -3/4$ ) and represents the static universe. Critical point (2) ( $\tilde{x} = 3/2, \tilde{\delta} = 3/2$ ) is a saddle and represents the scaling universe. Critical point (3) ( $\tilde{x} = 1, \tilde{\delta} = 0$ ) is an unstable node and represents the Einstein-de Sitter universe.

In addition, we are applying measurements of the Hubble parameter  $H(z)$  of galaxies from [35–37]. In this case the likelihood function is expressed by

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left( \frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2. \quad (6.4)$$

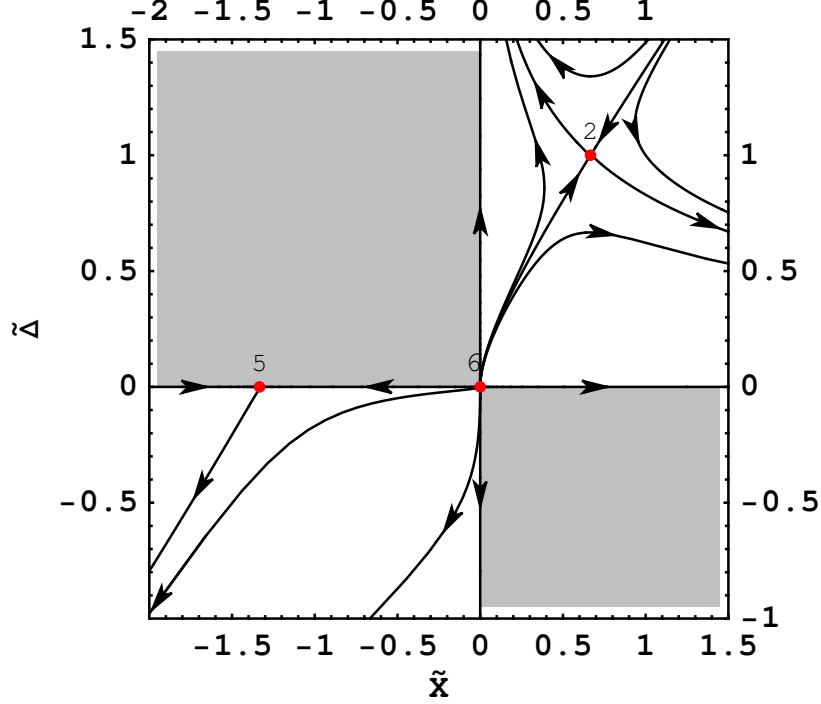
The final likelihood function is in the following form

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)}. \quad (6.5)$$

We use our own code CosmoDarkBox to estimate the model parameters. This code applies the Metropolis-Hastings algorithm [38, 39] and the dynamical system formulation of model dynamics to obtain the likelihood function [23, 40]. The dynamical system formulation of the cosmological dynamics developed in sec. 5 plays a crucial role in our method of estimation. We solve the system numerically using the Monte Carlo method and than put this solution to the corresponding expression for observables in our model.

For comparison models with diffusion with the  $\Lambda$ CDM model, we use Bayesian information criterion (BIC) [41, 42]. The BIC is defined as

$$\text{BIC} = -2 \ln L + j \ln n, \quad (6.6)$$



**Figure 3.** A phase portrait for dynamical system (5.24)-(5.25). Critical point (5) ( $\tilde{X} = -4/3$ ,  $\tilde{\Delta} = 0$ ) represents the static universe. Critical point (2) ( $\tilde{X} = 2/3$ ,  $\tilde{\Delta} = 1$ ) is a saddle and represents the scaling universe. Critical point (6) ( $\tilde{X} = 0$ ,  $\tilde{\Delta} = 0$ ) is an unstable node and represents the de Sitter universe. Note that if  $\tilde{\Delta} < 0$  the arrow of time indicates how the scale factor is decreasing during the evolution.

where  $L$  is the maximum of the likelihood function,  $j$  is the number of model parameters (in this paper for our models  $j = 3$  and for  $\Lambda$ CDM  $j = 2$ ) and  $n$  is number of data points (in this paper  $n = 622$ ).

## 6.2 Model of DM-DE interaction and $\tilde{w} = 0$

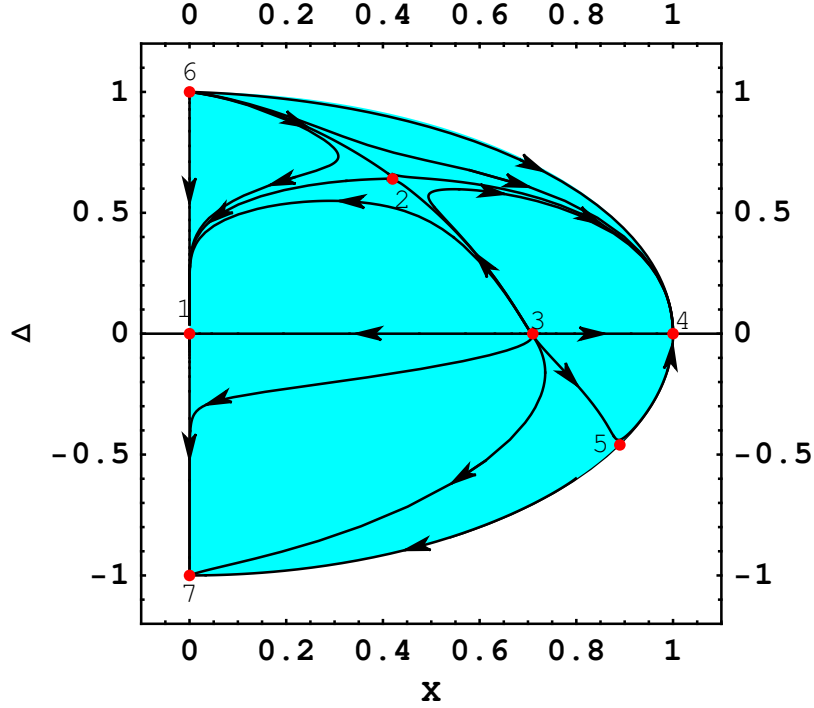
Let us consider the model of DM-DE interaction and with dark matter in the form of dust. We present a statistical analysis of the model parameters such as  $H_0$ ,  $\Omega_{\text{dm},0} = \frac{\rho_{\text{dm},0}}{3H_0^2}$ , where  $\rho_{\text{dm},0}$  is the present value of dark matter and  $\Omega_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T dt$ , where  $T$  is the present age of the Universe. We must have  $\Omega_{\gamma} \geq 0$  because  $\gamma \geq 0$  for a diffusion.

The Friedmann equation for  $\tilde{w} = 0$  in terms of the present values of the density parameters takes the form

$$\frac{H^2}{H_0^2} = \Omega_{\text{cm},0} a^{-3} + \frac{\Omega_{\gamma,0}}{\int^T dt} a^{-3} \int^t dt + \Omega_{\text{b},0} a^{-3} + \Omega_{\text{de}}(0) - \frac{\bar{\Omega}_{\gamma,0}}{\int^T a^{-3} dt} \int^t a^{-3} dt, \quad (6.7)$$

where  $\Omega_{\text{cm},0} = \frac{\rho_{\text{cm},0}}{3H_0^2}$ , where  $\rho_{\text{cm},0}$  is the present value of the conservative part of dark matter, which scales as  $a^{-3}$ ,  $\bar{\Omega}_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T a^{-3} dt$ .





**Figure 4.** A phase portrait for dynamical system (5.27)-(5.28). Critical point (1) represents the de Sitter universe. Critical point (2) is a saddle and represents the scaling universe. Critical point (3) is an unstable node and represents the Einstein-de Sitter universe. Critical point (4) represents the static universe. Critical point (5) represents the static universe. Critical point (6) is an unstable node and represents the de Sitter universe. Note that if  $\Delta < 0$  the arrow of time indicates how the scale factor is decreasing during the evolution.

In these estimation we use formulation of dynamics in the form of a two-dimensional non-autonomous system with the redshift variable  $z$ . This model possesses three parameters  $\gamma$ ,  $H_0$  and  $\Omega_m(z=0) = \Omega_{m,0}$

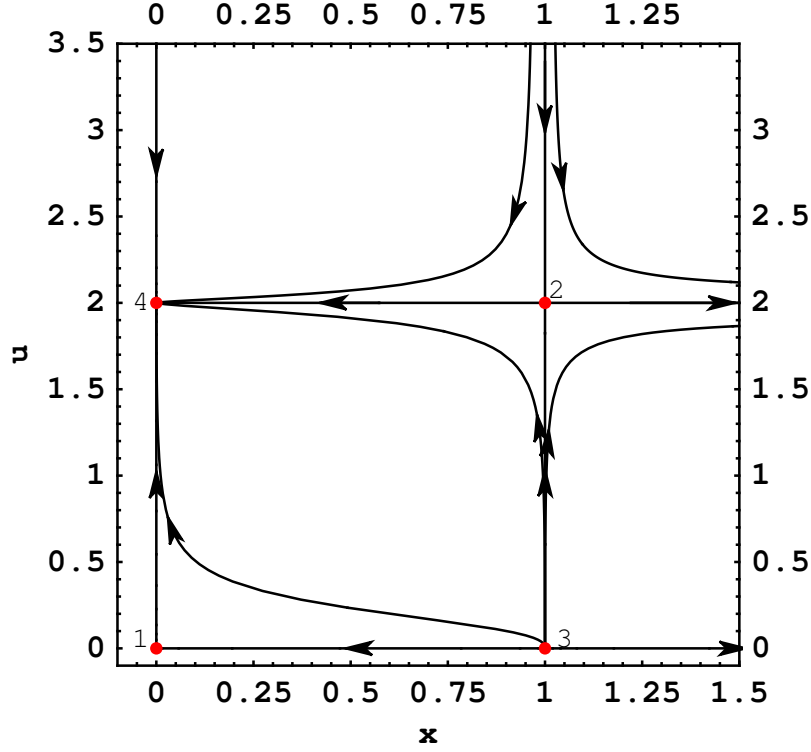
$$\Omega'_m = \frac{3}{1+z}\Omega_m - \frac{\gamma}{3H_0^3}P(1+z)^2, \quad (6.8)$$

$$P' = -\frac{3}{2}\frac{1}{1+z}\Omega_{m,0}P^3, \quad (6.9)$$

where  $P = \frac{H_0}{H}$  and  $' \equiv z$ .

Statistical results are presented in Table 3. Figure 9 shows the likelihood function with 68% and 95% confidence level projections on the plane  $(\Omega_{dm,0}, \Omega_\gamma)$ . For this case the value of reduced  $\chi^2$  is equal 0.187767.

The value of BIC, for this model is equal  $BIC_1=135.527$ . Because BIC for the  $\Lambda$ CDM model is equal  $BIC_{\Lambda CDM}=129.105$ ,  $\Delta BIC = BIC_1 - BIC_{\Lambda CDM}$  is equal 6.421. If that a value of  $\Delta BIC$  is more than 6, the evidence for the model is strong [42]. Consequently, the evidence in favor of the  $\Lambda$ CDM model is strong in comparison to our model.



**Figure 5.** A phase portrait for dynamical system (5.36)-(5.37). Critical point (1) ( $x = 0, u = 0$ ) represents the de Sitter universe without the diffusion effect. Critical point (2) ( $x = 1, u = 2$ ) is a saddle type and represents the scaling universe. Critical point (3) ( $x = 1, u = 0$ ) is an unstable node and represents the Einstein-de Sitter universe without the diffusion effect. The critical point (4) is representing the Einstein-de Sitter without the diffusion effect.

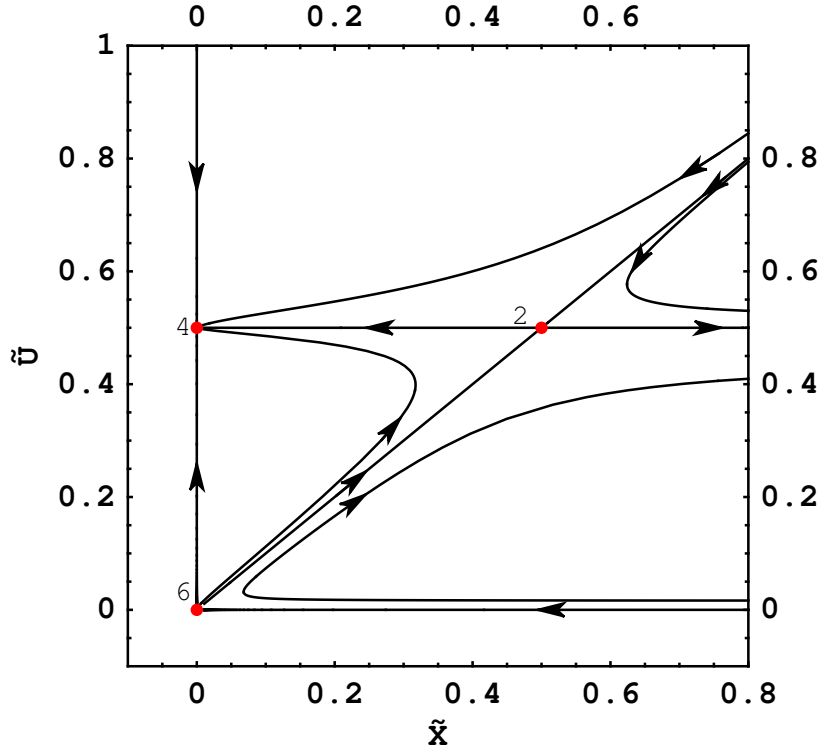
**Table 3.** The best fit and errors for the estimated model for SNIa+BAO+ $H(z)$ +AP test with  $H_0$  from the interval (65.0 (km/(s Mpc)), 71.0 (km/(s Mpc))),  $\Omega_{\text{dm},0}$  from the interval (0.25, 0.40),  $\Omega_{\gamma,0}$  from the interval (0.00, 0.20)  $\Omega_{\text{b},0}$  is assumed as 0.048468. The value of reduced  $\chi^2$  is equal 0.187767.

parameter	best fit	68% CL	95% CL
$H_0$	67.97 km/(s Mpc)	+0.75 -0.72	+1.57 -1.45
$\Omega_{\text{dm},0}$	0.2658	+0.0223 -0.0208	+0.0485 -0.0415
$\Omega_{\gamma,0}$	0.0135	+0.0735 -0.0135	+0.1570 -0.0135

### 6.3 Model with DM-DE interaction for $ma \rightarrow \infty$

Let us consider a late time behavior of the universe. For the case  $ma \rightarrow \infty$  we estimated values of cosmological parameters such as  $\Omega_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T a^2 dt$ ,  $\Omega_{Zm,0} = \frac{Zm}{3H_0^2}$ ,  $H_0$  and  $\gamma$ . The formula for the Friedmann equation in terms of the present values of the density parameters

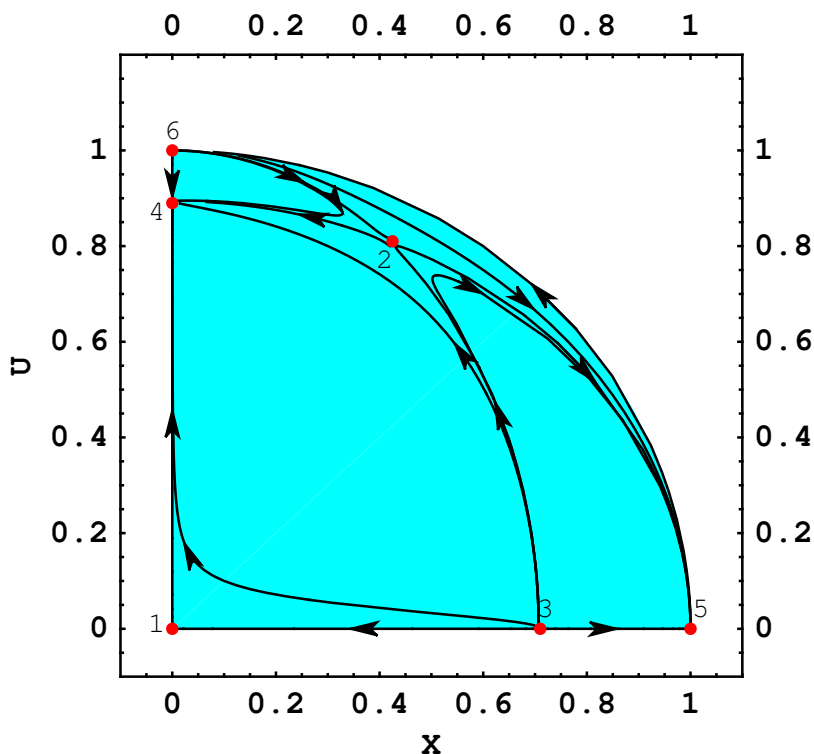




**Figure 7.** A phase portrait for dynamical system (5.40)-(5.41). Critical point (2) ( $\tilde{X} = 1/2, \tilde{U} = 1/2$ ) is a saddle and represents the scaling universe. Critical point (6) ( $\tilde{X} = 0, \tilde{U} = 0$ ) is an unstable node and represents the de Sitter universe. At this critical point the effect of diffusion are important. On the other hand the critical point (6) is an unstable stationary solution in which the effect of the non-zero term ( $Zm$ ) vanishes.

**Table 4.** The best fit and errors for the estimated model with  $w = 2/3$  for SNIa+BAO+ $H(z)$ +AP test with  $\Omega_{Zm,0}$  from the interval  $(0.22, 0.38)$ ,  $\Omega_{\gamma,0}$  from the interval  $(0.0, 0.03)$ ,  $\gamma$  from the interval  $(0.00(100 \text{ km}/(\text{s Mpc}))^3, 0.500(100 \text{ km}/(\text{s Mpc}))^3)$  and  $H_0$  from the interval  $(65.0 (\text{km}/(\text{s Mpc})), 71.0 (\text{km}/(\text{s Mpc})))$ .  $\Omega_{b,0}$  is assumed as 0.048468. The value of reduced  $\chi^2$  is equal 0.188201.

parameter	best fit	68% CL	95% CL
$H_0$	68.04	+0.73 −0.70	+1.27 −1.25
$\Omega_{\gamma,0}$	0.0106	+0.0082 −0.0106	+0.0137 −0.0106
$\Omega_{Zm,0}$	0.2943	+0.0356 −0.0077	+0.0536 −0.0231
$\gamma$	0.0299	+0.2198 −0.0299	+0.4555 −0.0299



**Figure 8.** A phase portrait for dynamical system (5.43)-(5.44). Critical point (1) represents the de Sitter universe without the diffusion effect. Critical point (2) is a saddle type and represents the scaling universe. Critical point (3) is an unstable node and represents the Einstein-de Sitter universe without the diffusion effect. The critical point (4) is representing the Einstein-de Sitter with the diffusion effect. Critical point (5) represents the static universe. Critical point (6) is an unstable node and represents the de Sitter universe.

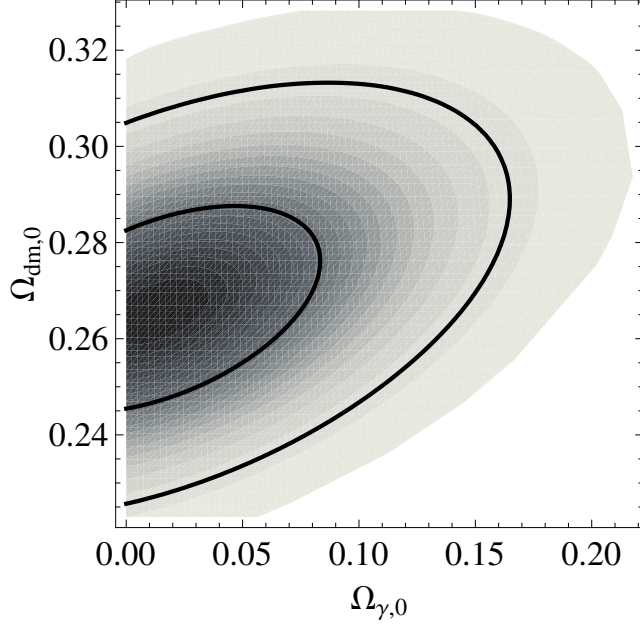
zero for the high redshifts (see Figure 11) and  $\Omega_{\text{de}}$  is negligible for the early universe. In consequence, we do not use the high redshift astronomical observations, such as CMB, to fit values of model parameters for our models.

## 7 Conclusion

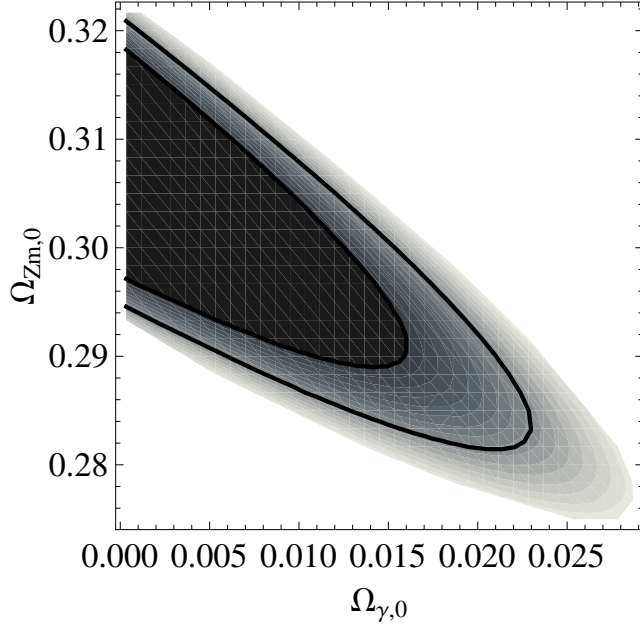
In this paper we studied the dynamics of DM-DE interaction with the relativistic diffusion process. For this aim we used the dynamical system methods, which enable us to study all evolutionary scenarios admissible for all initial conditions. We show that dynamics of our model reduces to the three-dimensional dynamical system, which in order is investigated on an invariant two-dimensional submanifold. From our dynamical analysis the dynamics is free from the difficulties, which are present in Alho et al.'s models with diffusion [12], namely there is no non-physical trajectories crossing the boundary set  $\rho_{\text{m}} = 0$  [15].

The model is tested by astronomical data in two cases of dark matter in the domain of low redshifts (SNIa, BAO,  $H(z)$  for galaxies and AP test).

In the model under consideration the energy density of dark matter is a growing function with the cosmological time on the cost of dark energy sector. In the basic formulas on  $H^2(z)$

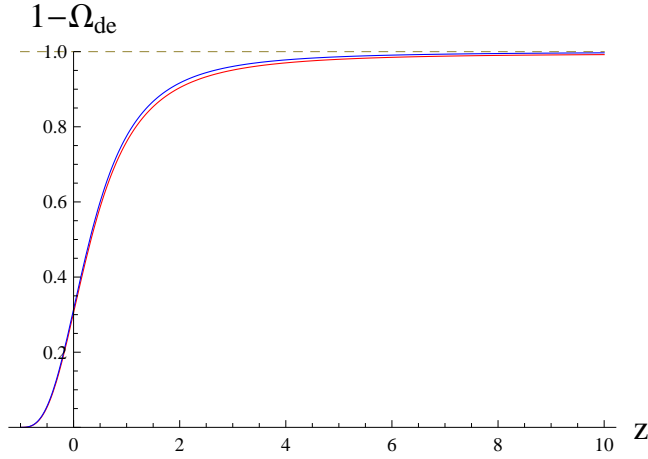


**Figure 9.** The intersection of the likelihood function of two model parameters  $(\Omega_{\text{dm},0}, \Omega_{\gamma,0})$ , for the case of the model of DM-DE interaction and  $\tilde{w} = 0$ , with the marked 68% and 95% confidence levels for SNIa+BAO+ $H(z)$ +AP test.  $\Omega_{\text{dm},0}$  is the present value of dark matter.



**Figure 10.** The intersection of the likelihood function of two model parameters  $(\Omega_{\text{Zm},0}, \Omega_{\gamma,0})$ , for the case of the model with DM-DE interaction for  $ma \rightarrow \infty$ , with the marked 68% and 95% confidence levels for SNIa+BAO+ $H(z)$ +AP test.

some additional terms appear related with the diffusion process itself. These contributions can be interpreted as the running Lambda term ( $\bar{\Omega}_{\gamma,0} \neq 0$ ) and a correction to the standard scaling law  $\propto a^{-3}$  for dark matter. At the present epoch the value of the density parameter



**Figure 11.** The diagram presents the evolution of  $1 - \Omega_{de}(z) = \frac{\Omega_{m,0}f(z)}{H^2(z)/H_0^2}$ , where  $z$  is redshift,  $\Omega_{m,0}f(z) = \frac{\rho_m(z)}{3H_0^2}$  and  $f(0) = 1$ , for the first model (blue line) and for the second model (red line). We assume the best fit values of model parameters (see Table 3 and 4). Note that, for the early universe, for the both models,  $1 - \Omega_{de}(z)$  is going to a constant (the horizontal asymptotics equals one). This means, that  $\Omega_{de}(z)$  for the high redshifts is negligible.

related with the dark matter correction is about 1% of total energy budget.

In the first model it is assumed dark matter in the form of dust. The estimated values of the model parameters are comparable with the parameters for the  $\Lambda$ CDM model and the value of reduced chi-square of this model is 0.187767. We also studied the second model with diffusion in a late time approximation:  $ma \rightarrow \infty$ . The value of density parameter of  $\Omega_{\gamma,0}$  related with diffusion is equal 0.0106. In this case the value of reduced chi-square is 0.188201. For comparison, the value of reduced chi-square of the  $\Lambda$ CDM model is 0.187483.

The value of  $\Delta\text{BIC} = \text{BIC}_i - \text{BIC}_{\Lambda\text{CDM}}$  for the first model is 6.421 and for the second model is equal 6.690. While the evidence is strong in favor of the  $\Lambda$ CDM model in comparison to our model, our model cannot be rejected based on our statistical analysis.

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